On double truncated (interval) WCRE and WCE

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Abstract

Measure of the weighted cumulative entropy about the predictability of failure time of a system have been introduced in [3]. Referring properties of doubly truncated (interval) cumulative residual and past entropy, several bounds and assertions are proposed in weighted version.

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1 Introduction. Interval weighted cumulative entropies

Let $x \in \mathbb{R}^+ \mapsto \varphi(x) \geq 0$ be a given measurable function. The weighted cumulative residual entropy (WCRE) $\mathcal{E}^{\mathrm{w}}_{\varphi}(X)$ and the weighted cumulative entropy (WCE) $\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(X)$ of a RV X with a cumulative distribution function (CDF) F and survival function (SF) $\overline{F}$ are defined by

$$\mathcal{E}^{\mathrm{w}}_{\varphi}(X) = \mathcal{E}^{\mathrm{w}}_{\varphi}(F) = -\int_{\mathbb{R}^+} \varphi(x)\overline{F}(x) \log \overline{F}(x) \mathrm{d}x, \quad \text{and} \tag{1.1}$$

$$\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(X) = \overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(F) = -\int_{\mathbb{R}^+} \varphi(x)F(x) \log F(x) \mathrm{d}x, \tag{1.2}$$

respectively. Assume that all integrals are absolutely convergent with the standard agreement $0 \log 0 = 0 \log \infty = 0$. Cf. [3], [1] and [6]. Further for more details and motivations see [8], [9].

For given pair of fixed values $(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ the CDF $F(x; t_1, t_2)$ and SF $\overline{F}(x; t_1, t_2)$ of a RV $X|t_1 < X < t_2$ take the forms

$$F(x; t_1, t_2) = \frac{F(x)}{F(t_2) - F(t_1)} \qquad \text{and} \qquad \overline{F}(x; t_1, t_2) = \frac{\overline{F}(x)}{\overline{F}(t_1) - \overline{F}(t_2)}. \tag{1.3}$$

We propose the following definition which we call the double truncated (interval) weighted cumulative residual entropy (IWCRE) $I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1, t_2)$ and the double truncated (interval) weighted cumulative entropy (IWCE) $I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2)$ of a RV $X|t_1 < X < t_2$:

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Definition 1.1 *Let (t_1, t_2) be a pair of fixed values in $\mathbb{R}^+ \times \mathbb{R}^+$. Using (1.3) define IWCRE of a RV $X|t_1 < X < t_2$ with SF $\overline{F}$ and WF φ by:*

$$\begin{aligned} I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1, t_2) &= -\int_{t_1}^{t_2} \varphi(x)\overline{F}(x; t_1, t_2) \log\, \overline{F}(x; t_1, t_2)\mathrm{d}x \\ &= -\int_{t_1}^{t_2} \varphi(x)\frac{\overline{F}(x)}{\overline{F}(t_1) - \overline{F}(t_2)} \log\, \frac{\overline{F}(x)}{\overline{F}(t_1) - \overline{F}(t_2)}\mathrm{d}x, \end{aligned} \tag{1.4}$$

and the IWCE of a RV $X|t_1 < X < t_2$ with CDF F is defined by

$$\begin{aligned} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) &= -\int_{t_1}^{t_2} \varphi(x)F(x; t_1, t_2) \log\, F(x; t_1, t_2)\mathrm{d}x \\ &= -\int_{t_1}^{t_2} \varphi(x)\frac{F(x)}{F(t_2) - F(t_1)} \log\, \frac{F(x)}{F(t_2) - F(t_1)}\mathrm{d}x. \end{aligned} \tag{1.5}$$

In particular $\varphi(x) \equiv 1$ the (1.4) and (1.5) yield the standard Interval cumulative residual entropy and the interval cumulative entropy, respectively. Cf. [2], [5], [4] and [7].

Passing to the limits $t_1 \to 0$ and $t_2 \to \infty$, the IWCRE (1.4) and IWCE (1.5) intend the WCRE (1.1) and the WCE (1.2), that is $I\mathcal{E}^{\mathrm{w}}_{\varphi}(0, \infty) = \mathcal{E}^{\mathrm{w}}_{\varphi}(X)$ and $I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(0, \infty) = \overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(X)$.

Remark 1.1 (a) *Owing to Definition 1.1, assume exponential RV X, $X \sim Exp(\lambda)$, $\lambda \geq 0$. In particular for given real constants $a_0, \ldots, a_n$ where $\varphi(x) = \sum_{i=0}^{n} a_i\, x^i \geq 0$. Set*

$$\gamma(b, z) = \int_0^z t^{b-1}\, e^{-t}\, \mathrm{d}t. \tag{1.6}$$

Following some straightforward computations one obtains

$$\begin{aligned} I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1, t_2) &= \Big(e^{-t_1/\lambda} - e^{-t_2/\lambda}\Big)^{-1} \sum_{i=0}^{n} a_i\, \lambda^{i+1}\Big\{\gamma(i+2, t_2/\lambda) - \gamma(i+2, t_1/\lambda)\Big\} \\ &+\Big(e^{-t_1/\lambda} - e^{-t_2/\lambda}\Big)^{-1} \log\Big(e^{-t_1/\lambda} - e^{-t_2/\lambda}\Big) \sum_{i=0}^{n} a_i\, \lambda^{i}\Big\{\gamma(i+1, t_2/\lambda) - \gamma(i+1, t_1/\lambda)\Big\}. \end{aligned} \tag{1.7}$$

(b) *More generally, let $\mu \in \mathbb{R}$, $\sigma > 0$, $\xi \in \mathbb{R}^+$ such that $\mu - \sigma/\xi \geq 0$ be location, scale and shape parameters respectively. Suppose that RV X has $GEV(\mu, \sigma, \xi)$ distribution, with CDF*

$$F_{GEV}(x) = e^{-y(x)}, \quad \textit{where} \quad y(x) = \big(1 + (\frac{x-\mu}{\sigma})\xi\big)^{-1/\xi}. \tag{1.8}$$

Moreover, set

$$\Pi_c(a, b) = \int_a^b y(t)^{c-1}e^{-y(t)}\mathrm{d}t, \quad a, b > 0, \ \ c \in \mathbb{R}. \tag{1.9}$$

If we assume $\varphi(x) = \sum_{i=0}^{n} b_i\, y(x)^i$, for $b_i \in \mathbb{R}$, $i = 0 \ldots n$ such that $\varphi(x) \geq 0$ with obvious motivations, the following expression is derived:

$$\begin{aligned} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) &= \Big(e^{-y(t_2)} - e^{-y(t_1)}\Big)^{-1} \sum_{i=0}^{n} b_i\Pi_{i+2}(t_1, t_2) \\ &+\Big(e^{-y(t_2)} - e^{-y(t_1)}\Big)^{-1} \log\Big(e^{-y(t_2)} - e^{-y(t_1)}\Big) \sum_{i=0}^{n} b_i\Pi_{i+1}(t_1, t_2). \end{aligned} \tag{1.10}$$

From now on for given WF φ we will use the notation $\psi(x)=\int_0^x \varphi(s)\mathrm{d}s$.

The following Lemma is straightforward.

Lemma 1.1 *For given a pari (t_1,t_2) and WF φ applying integrate by parts in Eqn (1.4) and (1.5) it can be written equivalent forms for IWCRE and IWCE:*

$$\begin{aligned} I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1,t_2) &= \frac{1}{\overline{F}(t_2)-\overline{F}(t_1)}\int_{t_1}^{t_2}\varphi(x)\overline{F}(x)\log\ \overline{F}(x)\mathrm{d}x+\bar{\delta}^{\mathrm{w}}_{\varphi}(t_1,t_2)\log\{\overline{F}(t_1)-\overline{F}(t_2)\} \\ &= \frac{1}{\overline{F}(t_2)-\overline{F}(t_1)}\int_{t_1}^{t_2}\varphi(x)\overline{F}(x)\log\ \overline{F}(x)\mathrm{d}x \\ &\quad+\left\{\frac{\psi(t_2)\overline{F}(t_2)-\psi(t_1)\overline{F}(t_1)}{\overline{F}(t_1)-\overline{F}(t_2)}+\mathbb{E}\big[\psi(X)|t_1<X<t_2\big]\right\}\log\{\overline{F}(t_1)-\overline{F}(t_2)\}, \end{aligned} \tag{1.11}$$

and in similar way:

$$\begin{aligned} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1,t_2) &= \frac{1}{F(t_1)-F(t_2)}\int_{t_1}^{t_2}\varphi(x)F(x)\log\ F(x)\mathrm{d}x+\delta^{\mathrm{w}}_{\varphi}(t_1,t_2)\log\{F(t_2)-F(t_1)\} \\ &= \frac{1}{F(t_1)-F(t_2)}\int_{t_1}^{t_2}\varphi(x)F(x)\log\ F(x)\mathrm{d}x \\ &\quad+\left\{\frac{\psi(t_2)F(t_2)-\psi(t_1)F(t_1)}{F(t_2)-F(t_1)}-\mathbb{E}\big[\psi(X)|t_1<X<t_2\big]\right\}\log\{F(t_2)-F(t_1)\}. \end{aligned} \tag{1.12}$$

Here

$$\bar{\delta}^{\mathrm{w}}_{\varphi}(t_1,t_2)=\int_{t_1}^{t_2}\varphi(x)\frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)}\mathrm{d}x \quad , \quad \delta^{\mathrm{w}}_{\varphi}(t_1,t_2)=\int_{t_1}^{t_2}\varphi(x)\frac{F(x)}{F(t_2)-F(t_1)}\mathrm{d}x. \tag{1.13}$$

Setting $\varphi'(x)$ the derivative function of WF $\varphi(x)$ with respect to x, $\varphi'(x)=\frac{\partial}{\partial x}\varphi(x)$ and following some standard calculations, we can write:

$$\begin{aligned} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1,t_2) &= \varphi(t_1)\overline{\mathcal{E}}_X(t_1,t_2)+\int_{t_1}^{t_2}\varphi'(x)\overline{\mathcal{B}}_X(x,t_2)\mathrm{d}x, \\ I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1,t_2) &= -\varphi(t_2)\overline{\mathcal{E}}_X(t_1,t_2)+\int_{t_1}^{t_2}\varphi'(y)\overline{\mathcal{B}}_X(t_1,y)\mathrm{d}y, \end{aligned} \tag{1.14}$$

here $\overline{\mathcal{E}}_X(t_1,t_2)$ represents the interval cumulative past entropy, denoted by $ICPE(X;t_1,t_2)$, in [2]. Moreover,

$$\begin{aligned} \overline{\mathcal{B}}_X(x,t_2) &= -\int_x^{t_2}\frac{F(y)}{F(t_2)-F(t_1)}\log\frac{F(y)}{F(t_2)-F(t_1)}\mathrm{d}y, \\ \overline{\mathcal{B}}_X(t_1,y) &= -\int_{t_1}^{y}\frac{F(x)}{F(t_2)-F(t_1)}\log\frac{F(x)}{F(t_2)-F(t_1)}\mathrm{d}x. \end{aligned} \tag{1.15}$$

In (1.14), substitute $\mathcal{E}_X(t_1,t_2)$ (denoted by $ICRE(X;t_1,t_2)$, cf. [2]) in $\overline{\mathcal{E}}_X(t_1,t_2)$, the analogue assertion for $I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1,t_2)$ holds.

Example 1.1 Let X be a RV from exponential distribution with mean $\frac{1}{\lambda}$, $\lambda>0$. According to the example in the end of [2]:

$$I\mathcal{E}(t_1,t_2)=\frac{1}{\lambda}+\frac{1}{\lambda}\log\left(1-e^{\lambda(t_1-t_2)}\right)+\frac{(t_2-t_1)e^{\lambda t_1}}{e^{\lambda t_1}-e^{\lambda t_2}},\ \ t_2>t_1\geq 0. \tag{1.16}$$

We observe that for fixed value $t_2 \in (0, \infty)$, (1.16) is decreasing in $t_1 \in (0, \infty)$. Now, assume the WF $\varphi(x) = e^{\alpha x}$, $\alpha < \lambda$, applying (1.4) yields the following expression:

$$\begin{aligned} I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1, t_2) &= \frac{1}{(\lambda - \alpha)(e^{-\lambda t_2} - e^{-\lambda t_1})} . \Big\{ \lambda \big(t_2 e^{(\alpha-\lambda)t_2} - t_1 e^{(\alpha-\lambda)t_1} \big) \\ &+ \frac{\lambda}{(\alpha - \lambda)} . \big(e^{(\alpha-\lambda)t_2} - e^{(\alpha-\lambda)t_1} \big) + \big(e^{(\alpha-\lambda)t_2} - e^{(\alpha-\lambda)t_1} \big) . \log \big(e^{-\lambda t_1} - e^{-\lambda t_2} \big) \Big\}. \end{aligned} \tag{1.17}$$

Note that when $\alpha \to 0$ then $I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1, t_2) \to I\mathcal{E}(t_1, t_2)$. Applying mathematical software such as Maple, one can easily check that for given all λ, α, (1.17) is not monotonic decreasing in t_1. This means, if the monotonicity property for ICRE is fulfilled then there is no guarantee IWCRE is monotonic as well.

2 Bounds for the IWCE and IWCRE

In this section , we give several bounds for the IWCRE and IWCE by using assertions established in Section 1. Let us start with an alternative representation for the IWCRE and IWCE. In fact it follows the same line as (1.11) and (1.12) but is more elementary.

Let X be a non-negative RV, moreover consider a pair $(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Set

$$\begin{aligned} \gamma(t_1, t_2) &= -\int_{t_1}^{t_2} \varphi(x)\ F(x) \log F(x)\ \mathrm{d}x, \\ \vartheta_i(t_1, t_2) &= \frac{F(t_i)}{F(t_2) - F(t_1)}, \quad i = 1, 2. \end{aligned}$$

therefore, we can write

$$\begin{aligned} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) = &- \int_{t_1}^{t_2} \varphi(x) \frac{F(x)}{F(t_2) - F(t_1)} \log\ \vartheta_1(x, t_2) \mathrm{d}x \\ &- \int_{t_1}^{t_2} \varphi(x) \frac{F(x)}{F(t_2) - F(t_1)} \log\ \frac{F(t_2) - F(x)}{F(t_2) - F(t_1)} \mathrm{d}x, \end{aligned} \tag{2.1}$$

in addition,

$$\begin{aligned} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) = &- \int_{t_1}^{t_2} \varphi(x) \frac{F(x)}{F(t_2) - F(t_1)} \log\ \vartheta_2(t_1, x) \mathrm{d}x \\ &- \int_{t_1}^{t_2} \varphi(x) \frac{F(x)}{F(t_2) - F(t_1)} \log\ \frac{F(x) - F(t_1)}{F(t_2) - F(t_1)} \mathrm{d}x. \end{aligned} \tag{2.2}$$

For given pair (t_1, t_2) define functions $\bar{\gamma}_1$ and $\bar{\gamma}_2$ in terms of $\overline{F}(x)$ in a similar fashion, then analogue formulas take place for IWCRE as well.
Now we are in the position to establish Theorem 2.1 below. Recalling (1.13), (2.1), (2.2) and using the inequality $\log(1 - s) \geq s/(s - 1)$, $0 < s < 1$ we provide lower bounds for the IWCE, omitting the proof.

Theorem 2.1 *Let X be a non-negarive RV, with CDF F. Then given WF $x \in \mathbb{R}^+ \mapsto \varphi(x) \geq 0$ obeys*

$$\begin{aligned} &I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) \geq \\ &\quad \big(F(t_2) - F(t_1)\big)^{-1} \Big[\gamma(t_1, t_2) + F(t_2) \big(\psi(t_2) - \psi(t_1) \big) \Big] + \delta^{\mathrm{w}}_{\varphi}(t_1, t_2) \Big(1 + \log F(t_1) \Big). \end{aligned} \tag{2.3}$$

It is worth noting that in a similar manner by owing to the definition of $\bar{\delta}_\varphi^{\rm w}(t_1,t_2)$ *in (1.13), if we swap* γ *and* $\overline{\gamma}$*, also* F *and* $\overline{F}$ *in 2.3 we get analogue lower bounds for* $I\mathcal{E}_\varphi^{\rm w}(t_1,t_2)$*, where*

$$\overline{\gamma}(t_1,t_2) = -\int_{t_1}^{t_2} \varphi(x)\, \overline{F}(x) \log \overline{F}(x)\, {\rm d}x.$$

An immediate application of Theorem 2.1 follows.

Proposition 2.1 *Consider function* $g(\varepsilon)$ *in a form as*

$$g(\varepsilon) = \big(1 + (\frac{\varepsilon-\mu}{\sigma})\xi\big)^{-1/\xi}, \quad \sigma > 0,\ \mu \in \mathbb{R},\ \xi \in \mathbb{R}^+,\ \mu - \sigma/\xi \geq 0.$$

Then for constant $0 \leq x < y \leq \infty$ *and* θ_i*,* $i = 0 \ldots n$ *the inequality*

$$\Big(g(x) - 1 + \log\big(e^{-g(y)} - e^{-g(x)}\big)\Big) \sum_{i=0}^n \theta_i\, \Pi_{i+1}(x,y) \geq e^{-g(y)} \sum_{i=0}^n \theta_i \int_x^y g(s)^i\, {\rm d}s. \tag{2.4}$$

holds true. Here Π *stands as before in (1.9):*

$$\Pi_c(a,b) = \int_a^b y(t)^{c-1} e^{-y(t)} {\rm d}t, \quad a,b > 0, \ \ c \in \mathbb{R}.$$

Theorem 2.2 *Suppose that* X *is a RV with CDF* F *and finite* $I\overline{\mathcal{E}}_\varphi^{\rm w}(t_1,t_2)$*. Given WF* φ*, set*

$$\eta(X) = \frac{1}{F(x)} \int_0^x \varphi(y) F(y) {\rm d}y$$

Then

$$I\overline{\mathcal{E}}_\varphi^{\rm w}(t_1,t_2) \leq \mathbb{E}\left[\eta(X)|t_1 \leq X \leq t_2\right].$$

Proof. First we begin from the expression $\eta(X)$:

$$\mathbb{E}\left[\eta(X)|t_1 \leq X \leq t_2\right] = \int_{t_1}^{t_2} \left(\int_0^x \varphi(y) \frac{F(y)}{F(x)} {\rm d}y\right) \frac{f(x)}{F(t_2) - F(t_1)} {\rm d}x$$
$$= \int_0^{t_1} \left(\int_{t_1}^{t_2} \frac{f(x)}{F(x)} {\rm d}x\right) \varphi(y) \frac{F(y)}{F(t_2)-F(t_1)} {\rm d}y + \int_{t_1}^{t_2} \left(\int_y^{t_2} \frac{f(x)}{F(x)} {\rm d}x\right) \varphi(y) \frac{F(y)}{F(t_2)-F(t_1)} {\rm d}y.$$

Further using the relation $\displaystyle\int_a^b \frac{f(x)}{F(x)} {\rm d}x = \log F(b) - \log F(a)$ leads

$$\begin{aligned} \mathbb{E}\Big[\eta(X)|t_1 \leq X \leq t_2\Big] &= \int_0^{t_1} \big[\log F(t_2) - \log F(t_1)\big] \varphi(y) \frac{F(y)}{F(t_2)-F(t_1)} {\rm d}y \\ &+ \int_{t_1}^{t_2} \big[\log F(t_2) - \log F(y)\big] \varphi(y) \frac{F(y)}{F(t_2)-F(t_1)} {\rm d}y \\ &\geq \int_{t_1}^{t_2} \big[\log\{F(t_2) - F(t_1)\} - \log F(y)\big] \varphi(y) \frac{F(y)}{F(t_2)-F(t_1)} {\rm d}y. \end{aligned} \tag{2.5}$$

In the last line of (2.5) the inequality holds from $\log F(t_2) - \log F(t_1) \geq 0$. For given $t_1 < t_2 \in \mathbb{R}^+$ we also know $\log F(t_2) \geq \log\big[F(t_2) - F(t_1)\big]$. This completes the proof. ■

Remarkably observe that, IWCRE possesses the similar property in Theorem 2.2, hence we can write:

$$I\mathcal{E}_{\varphi}^{\mathrm{w}}(t_1,t_2) \leq \mathbb{E}\left[\bar{\eta}(X)|t_1 \leq X \leq t_2\right],$$

where $\bar{\eta}(x) = \dfrac{1}{\overline{F}(x)} \displaystyle\int_x^{\infty} \varphi(y)\overline{F}(y)\mathrm{d}y.$

The next theorem extends the result of Theorem 8 from [2]. Here we set

$$IH(X;t_1,t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)}{F(t_2)-F(t_1)} \mathrm{d}x,$$

Note that $IH(X;t_1,t_2)$ is an extension of Shannon entropy based on a doubly truncated (interval) RV, see [7].

Theorem 2.3 *Let X be a non-negative continuous RV with PDF and CDF respectively $f(x)$ and $F(x)$, then for give WF $\varphi(x)$,*

$$I\overline{\mathcal{E}}_{\varphi}^{\mathrm{w}}(t_1,t_2) \geq \alpha(t_1,t_2).\exp\{IH(X;t_1,t_2)\}.$$

Here

$$\alpha(t_1,t_2) = \exp\Big\{ \int_{\beta_1}^{\beta_2} \log\big[u\, \varphi(F^{-1}\{uF(t_2)-uF(t_1)\})|\log u|\big]\mathrm{d}u\Big\},$$

where for $i=1,2$, $\beta_i = \dfrac{F(t_i)}{F(t_2)-F(t_1)}$.

Proof. The proof follows directly from the Log-Sum inequality while implies

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)}{F(t_2)-F(t_1)} \mathrm{d}x \\
&\quad - \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log\big[\varphi(x)\frac{F(x)}{F(t_2)-F(t_1)}|\log \frac{F(x)}{F(t_2)-F(t_1)}|\big]\mathrm{d}x \\
&\quad \geq -\log \int_{t_1}^{t_2} \varphi(x)\frac{F(x)}{F(t_2)-F(t_1)}|\log \frac{F(x)}{F(t_2)-F(t_1)}|\mathrm{d}x \\
&\quad = \log \frac{1}{I\overline{\mathcal{E}}_{\varphi}^{\mathrm{w}}(t_1,t_2)}.
\end{aligned}$$

■

Remark 2.1 *The similar arguments for IWCRE is achieved. In other words, owing to the definition of $IH(X;t_1,t_2)$ we have*

$$I\mathcal{E}_{\varphi}^{\mathrm{w}}(t_1,t_2) \geq \bar{\alpha}(t_1,t_2).\exp\Big\{IH(X;t_1,t_2)\Big\}.$$

Here

$$\bar{\alpha}(t_1,t_2) = \exp\Big\{ \int_{\kappa_1}^{\kappa_2} \log\big[u\, \varphi(\overline{F}^{-1}\{u\overline{F}(t_1)-u\overline{F}(t_2)\})|\log u|\big]\mathrm{d}u\Big\},$$

where for $i=1,2$, $\kappa_i = \dfrac{\overline{F}(t_i)}{\overline{F}(t_1)-\overline{F}(t_2)}$.

In Theorem 2.4 below (cf. Theorem 2.3, [2]), let $\bar{\lambda}(x) = \dfrac{f(x)}{F(x)}$ be reversed failure rate function and $h_2(t_1, t_2)$ denotes the generalized failure rate (GFR) by virtue of the doubly truncated RV, defined in [5]. Assume also $\varphi(x)$ be a positive WF on an open domain with $\psi(x) = \int_0^x \varphi(s)\mathrm{d}s$ and set $\mathcal{M}(t_1, t_2) = \mathbb{E}\Big[\psi(t_2) - \psi(X)|t_1 \le X \le t_2\Big]$. Then the next theorem is provided:

Theorem 2.4 *The IWCE is an increasing function in t_2 iff for all given $(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, $t_1 < t_2$:*

$$\begin{aligned} &I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) \\ &\quad \le \mathcal{M}(t_1, t_2) + (\psi(t_2) - \psi(t_1))\frac{F(t_1)}{F(t_2) - F(t_1)} - \varphi(t_2)[\bar{\lambda}(t_2)]^{-1} \log \frac{F(t_2)}{F(t_2) - F(t_1)}. \end{aligned} \tag{2.6}$$

Proof. According to the form (1.12), differentiating IWCE with respect to t_2 yields

$$\begin{aligned} \frac{\partial}{\partial t_2} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) &= \frac{f(t_2)}{[F(t_2) - F(t_1)]^2} \int_{t_1}^{t_2} \varphi(x) F(x) \log F(x) \mathrm{d}x - \frac{\varphi(t_2) F(t_2) \log F(t_2)}{F(t_2) - F(t_1)} \\ &+ \frac{f(t_2)}{F(t_2) - F(t_1)} \left[\mathcal{M}(t_1, t_2) + (\psi(t_2) - \psi(t_1)) \frac{F(t_1)}{F(t_2) - F(t_1)}\right] \\ &+ \left(\frac{\partial}{\partial t_2}\mathcal{M}(t_1, t_2) + \frac{\varphi(t_2) F(t_1)}{F(t_2) - F(t_1)} - \frac{f(t_2) F(t_1)(\psi(t_2) - \psi(t_1))}{[F(t_2) - F(t_1)]^2}\right) \log\{F(t_2) - F(t_1)\}. \end{aligned} \tag{2.7}$$

Furthermore differentiating the $\mathcal{M}(t_1, t_2)$ with respect to t_2 implies

$$\frac{\partial}{\partial t_2}\mathcal{M}(t_1, t_2) = \varphi(t_2) - \mathcal{M}(t_1, t_2) h_2(t_1, t_2). \tag{2.8}$$

After that substitute (2.8) in (2.7), we have

$$\begin{aligned} &\frac{\partial}{\partial t_2} I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) \\ &\quad = h_2(t_1, t_2). \Bigg[\mathcal{M}(t_1, t_2) - I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1, t_2) + (\psi(t_2) - \psi(t_1)) \frac{F(t_1)}{F(t_2) - F(t_1)} \\ &\qquad - \varphi(t_2)[\bar{\lambda}(t_2)]^{-1} \log \frac{F(t_2)}{F(t_2) - F(t_1)}\Bigg]. \end{aligned}$$

The inequality (2.6) then follows. ■

Theorem 2.5 (Cf. [2] Theorem 2.10) *Suppose X and Y are two non-negative, iid RVs with SF $\overline{F}$. Then for given WF φ , consequently ψ and $(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, $t_1 < t_2$:*

$$\begin{aligned} &\mathbb{E}\Big(|\psi(X) - \psi(Y)||t_1 \le X \le t_2, t_1 \le Y \le t_2\Big) \\ &\le \frac{2I\mathcal{E}^{\mathrm{w}}_{\varphi}(t_1, t_2)}{\overline{F}(t_1) - \overline{F}(t_2)} - \frac{\log[\overline{F}(t_1) - \overline{F}(t_2)]}{\overline{F}(t_1) - \overline{F}(t_2)} \left(\overline{\mathcal{M}}(t_1, t_2) + (\psi(t_2) - \psi(t_1)) \frac{\overline{F}(t_2)}{\overline{F}(t_1) - \overline{F}(t_2)}\right). \end{aligned} \tag{2.9}$$

Here

$$\overline{\mathcal{M}}(t_1, t_2) = \mathbb{E}\Big[\psi(X) - \psi(t_1)|t_1 \le X \le t_2\Big].$$

Proof. Following the similar arguments in Theorem 2.10, [2], for two iid RVs X and Y we have

$$\begin{aligned}
&2\,\frac{\overline{F}(u)}{\overline{F}(t_1)-\overline{F}(t_2)}-2\left(\frac{\overline{F}(u)}{\overline{F}(t_1)-\overline{F}(t_2)}\right)^2\\
&\quad=P\{\max(\varphi(X),\varphi(Y))>u|t_1\le X\le t_2,t_1\le Y\le t_2\}\\
&\quad\quad-P\{\min(\varphi(X),\varphi(Y))>u|t_1\le X\le t_2,t_1\le Y\le t_2\}.
\end{aligned}\tag{2.10}$$

By multiplying the both sides of (2.10) in $\varphi(u)$ and then integrating from t_1 to t_2, we obtain

$$\begin{aligned}
&\frac{2}{[\overline{F}(t_1)-\overline{F}(t_2)]^2}\int_{t_1}^{t_2}\varphi(u)\overline{F}(u)\left[\overline{F}(t_1)-\overline{F}(t_2)-\overline{F}(u)\right]\mathrm{d}u\\
&\quad=\mathbb{E}\Big(|\psi(X)-\psi(Y)||t_1\le X\le t_2,t_1\le Y\le t_2\Big).
\end{aligned}$$

At this stage we apply the non-decreasing property for ψ in x and deduce that for all $x\in(0,1)$ and $b\in(0,1)$, $x(b-x)\le x|\log x|$. This leads to

$$\begin{aligned}
&\mathbb{E}\Big(|\psi(X)-\psi(Y)||t_1\le X\le t_2,t_1\le Y\le t_2\Big)\\
&\quad\le\frac{2}{[\overline{F}(t_1)-\overline{F}(t_2)]^2}\int_{t_1}^{t_2}\varphi(u)\overline{F}(u)|\log\overline{F}(u)|\mathrm{d}u.
\end{aligned}\tag{2.11}$$

Combining (2.11) and (1.11) the assertion (2.9) clarifies. ■

Remark 2.2 *It can be observed explicitly that the LHS of inequality (2.9) in Theorem 2.5 is bigger and equal than:*

$$\mathbb{E}\Big(|\psi(X)-\mathbb{E}(\psi(X))||t_1\le X\le t_2\Big).$$

Moreover, similar inequalities as (2.9) for IWCE can be hold:

$$\begin{aligned}
&\mathbb{E}\Big(|\psi(X)-\psi(Y)||t_1\le X\le t_2,t_1\le Y\le t_2\Big)\\
&\le\frac{2I\overline{\mathcal{E}}^{\mathrm{w}}_{\varphi}(t_1,t_2)}{F(t_2)-F(t_1)}-\frac{\log[F(t_2)-F(t_1)]}{F(t_2)-F(t_1)}\left(\mathcal{M}(t_1,t_2)+(\psi(t_2)-\psi(t_1))\frac{F(t_1)}{F(t_2)-F(t_1)}\right).
\end{aligned}\tag{2.12}$$

Here

$$\mathcal{M}(t_1,t_2)=\mathbb{E}\Big[\psi(t_2)-\psi(X)|t_1\le X\le t_2\Big].$$

.

We conclude the paper by using Theorem 2.3 for uniform RV, Theorem 2.5 in exponential form and WF $\varphi(x)=\sum\limits_{i=0}^{n}a_ix^i$, $a_i\in\mathbb{R}$, $\varphi(x)\ge0$, recall also (1.7, in order to explore some emerged inequalities.

Corollary 2.1 (i) *For constant $0\le a<b\le1$. Assume arbitrary function $f:\mathbb{R}\mapsto\mathbb{R}^+$ we get*

$$\int_a^b s\,f(s)\log\frac{b-a}{s}\,\mathrm{d}s\ge(b-a)\exp\int_a^b\log\left[s\,f(s)|\log\frac{b-a}{s}|\right]\frac{\mathrm{d}s}{b-a}.$$

(ii) *Consider constant* $0 \le a < b \le \infty$, $c, p \in \mathbb{R}^+$. *Further set*

$$\overline{\gamma}_p(a,b) = \int_a^b t^{p-1}e^{-t}\,\mathrm{d}t = \gamma(p,b) - \gamma(p,a), \quad \textit{by virtue of (1.6)}$$
$$\Delta_c(a,b) = e^{-a/c} - e^{-b/c}.$$

Then constants $\varepsilon_0, \dots, \varepsilon_n$, *such that* $\sum_{i=0}^{n} \varepsilon_i x^i \ge 0$, $x \in \mathbb{R}^+$ *are satisfied in the inequality:*

$$\begin{aligned} 2\sum_{i=0}^{n} \big(c.\Delta_c(a,b) - \log\Delta_c(a,b)\big)c^i\, \varepsilon_i\, \overline{\gamma}_{i+1}(a/c, b/c) - \sum_{i=0}^{n} \varepsilon_i\, c^{i+1}2^{-i}\overline{\gamma}_{i+1}(2a/c, 2b/c) \\ \le \sum_{i=0}^{n}\big(2 - \log\Delta_c(a,b)/(i+1)\big)\varepsilon_i\, c^{i+1}\overline{\gamma}_{i+2}(a/c,b/c) \\ + \log\Delta_c(a,b)\Big\{\sum_{i=0}^{n}\frac{\varepsilon_i}{i+1}\big(a^{i+1}e^{-a/c} - b^{i+1}e^{-b/c}\big)\Big\}. \end{aligned} \tag{2.13}$$

Note that in special case $c = 1$, $a \to 0$ *and* $b \to \infty$, *final inequality (2.13) takes the form*

$$\sum_{i=0}^{n} \varepsilon_i\, \Gamma(i+1)(1 - 2^{-i-1}) \le \sum_{i=0}^{n} \varepsilon_i\, \Gamma(i+2).$$

Here $\Gamma(.) = \overline{\gamma}_.(0,\infty)$ *refers to Gamma function.*

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